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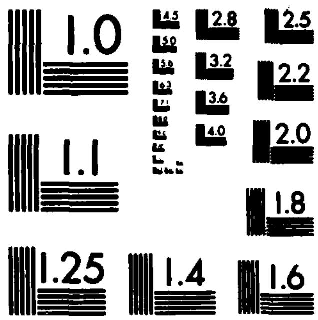

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DAVID W. TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Maryland 20084

FEASIBILITY OF APPLYING THE FEARS PROGRAM TO THE PLATE BENDING PROBLEM

by

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Contract N00167-82-M-0743

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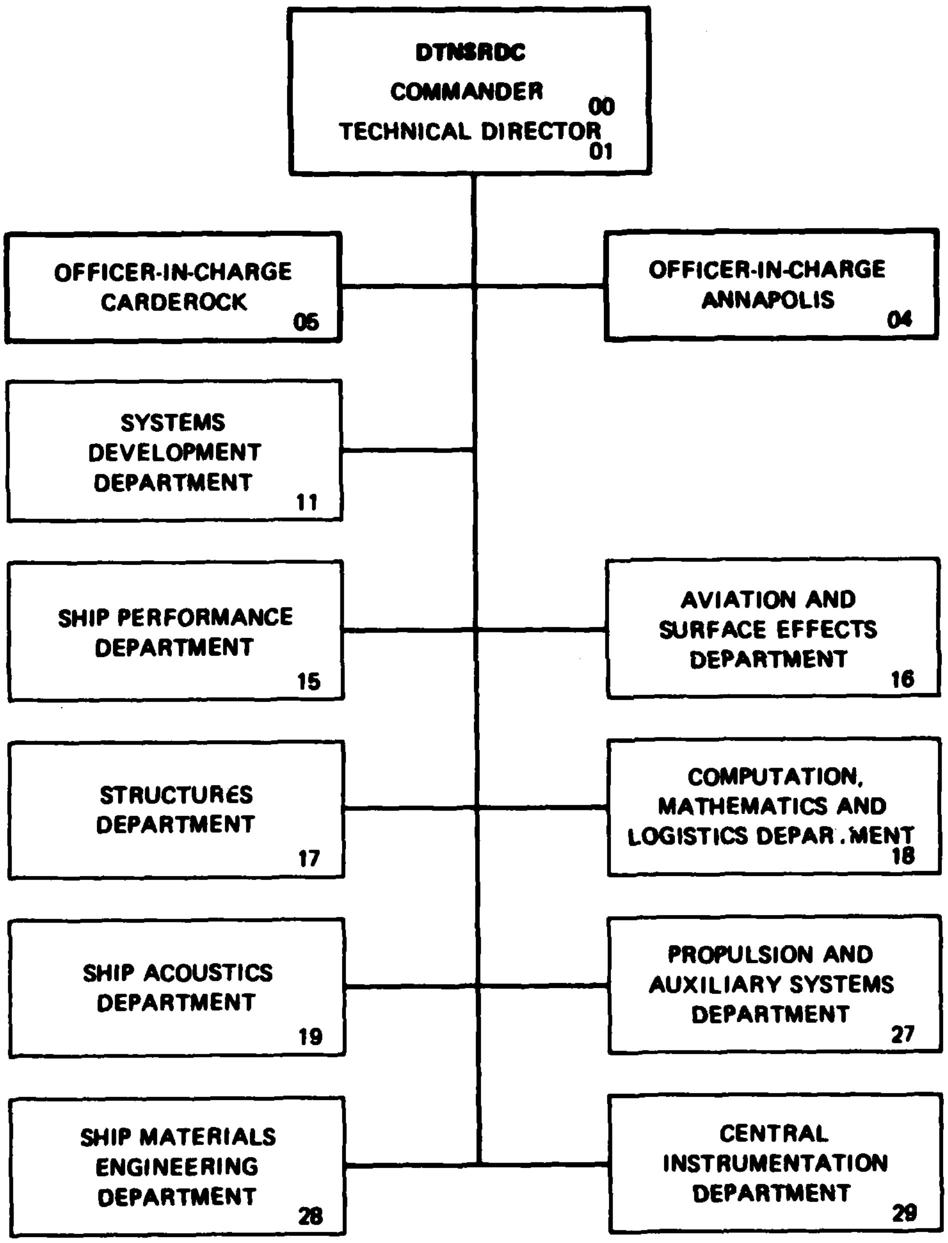
COMPUTATION, MATHEMATICS, AND LOGISTICS DEPARTMENT
DEPARTMENTAL REPORT

May 1983 DTNSRDC/CMLD-83/11

MAJOR DTNSRDC ORGANIZATIONAL COMPONENTS



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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: DTNSRDC/CMLD-83/11	2. GOVT ACCESSION NO. A129395
	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): FEASIBILITY OF APPLYING THE FEARS PROGRAM TO THE PLATE BENDING PROBLEM	5. TYPE OF REPORT & PERIOD COVERED
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Donald A. Gignac Ivo Babuska	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: David Taylor Naval Ship Research and Development Center Bethesda, Maryland 20084	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Program Element 61152 Task Area ZR0140201 Work Unit 1844-140
11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE: May 1983
	13. NUMBER OF PAGES: 27
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Finite Element Method
Plate Bending Problem
FEARS Program
Mesh Refinement
Error Estimation
Adaptive Meshing
Lame' Equations
Splitting Method
Biharmonic Equation
Moments
Shear Forces

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report documents an investigation of the feasibility of the application of the Finite Element Adaptive Research Solver (FEARS) computer program to the Plate Bending Problem. Two methods of reducing this biharmonic problem to an elliptic system of two second order partial differential equations are considered. The first is the splitting method and the second is the transformation to the Lame' system of elasticity equations. The FEARS program is used to solve these reduced systems for three examples. The

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ABSTRACT

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ADMINISTRATIVE INFORMATION

This work was performed under David W. Taylor Naval Ship R&D Center's (DTNSRDC) Independent Research Program, Program Element 61152, Task Area ZR0140201, DTNSRDC Work Unit 1844-140. Professor Ivo Babuska of the Institute for Physical Science and Technology, University of Maryland (Contract N00167-82-M-0743) is a contractor to DTNSRDC.

INTRODUCTION

The Finite Element Adaptive Research Solver (FEARS) computer program is an adaptive finite element solver with a posteriori error estimates.[1,2*] It solves boundary value problems for an elliptic system of two partial differential equations with two independent variables, using first degree elements which have continuous displacements across element boundaries. Elements with such continuous displacements are said to be C^0.

The classical plate bending problem is described by a partial differential equation of the fourth order, the biharmonic equation. It should be possible to transform this biharmonic problem into an elliptic system of two second

* A complete list of references is given on page 21.

order partial differential equations, but the problem of obtaining a family of such transformations has not been solved yet for the general case.

The biharmonic problem can be solved through a reduction to a system of second order equations by various methods: the Reissner-Mindlin refined plate formulation [3,4] which provides three fields of variables (displacements and two rotations), the Hermann-Mioshi method[5,6,7,8,9] and the Hermann-Johnson method[5,6,10] which provides a system of four equations. Other methods of solving biharmonic equations are described by Scholz.[11] In addition, the splitting method first considered by Glowinski[12] and Mercier[13] and further developed by others[8,9,11,14,15] provides a system of two second order equations. Finally the biharmonic problem can be transformed into the Lame' system of elasticity equations by exploiting the fact that the Airy function is biharmonic. The last two approaches provide a boundary value problem solvable by the FEARS program.

This report studies the performance of these two last approaches with respect to the FEARS program and assesses their advantages and disadvantages. The second section contains basic information about the FEARS program. The sample problem, the method of transformation to the Lame' equations, and the splitting method are described in the third section. The numerical experiments are described and some conclusions and recommendations are presented in the fourth and fifth sections respectively.

ADAPTIVE FINITE ELEMENT SOLVER FEARS

The FEARS program has been described in two informal reports.[1,2] It solves boundary value problems for linear self-adjoint elliptic systems of partial differential equations of second order with two field variables in two dimensions. It also provides error estimates for the finite element solution with respect to the unknown exact solution of the problem.

FEARS deals with domains which are the union of the interiors of a small number of open curvilinear rectangles bounded by circular arcs or straight line segments. These curvilinear rectangles are called 2-D domains and are denoted by D_j^2, $j=1,\dots,N_2$. The open arcs are called 1-D domains and are denoted by D_j^1, $j=1,\dots,N_1$. The vertices of the 2-D domains are called 0-D domains and are denoted by D_j^0, $j=1,\dots,N_0$.

The FEARS program solves problems described in the weak form

$$\sum_i \int_{D_i^2} \left\{ \left[\frac{\partial V}{\partial Z}\right]^T A_i \left[\frac{\partial U}{\partial Z}\right] + V^T B_i \left[\frac{\partial U}{\partial Z}\right] + \left[\frac{\partial V}{\partial Z}\right]^T B_i^T U + V^T C_i U \right\} dx_1 dx_2$$

$$+ \sum_j \int_{D_j^1} V^T \gamma_j U \, ds =$$

$$\sum_i \int_{D_i^2} \left\{ \left[\frac{\partial V}{\partial Z}\right]^T D(x_1 x_2) + V^T E(x_1 x_2) \right\} dx_1 dx_2 \tag{1}$$

$$+ \sum_j \int_{D_j^1} \varepsilon_j(s) V ds$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and

$$\frac{\partial U}{\partial Z} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

V and $\frac{\partial V}{\partial Z}$ are defined similarly.

Further

A_i is a 4 x 4 symmetric (constant) matrix,

B_i is a 2 x 4 (constant) matrix,

C_i is a 2 x 2 symmetric (constant) matrix,

γ_j is a 2 x 2 symmetric (constant) matrix,

$D_i(x_1,x_2)$ is a 4 x 1 vector valued function with components $S_k(x_1,x_2)$ $(k = 1,2,3,4)$,

$E_i(x_1,x_2)$ is a 2 x 1 vector valued function,

$\varepsilon_i(s)$ (where s is arc length) is a 1 x 2 vector valued function.

The trial space used for U and the test space used for V allow the imposition of both essential and natural boundary conditions. For a discussion of the physical significance of the matrices with examples, the reader is referred to the User's Guide.[2]

The admissible error norm for the a posteriori estimate of the error $e = u_{FE} - u_o$ (where u_{FE} and u_o denote the finite element and exact solutions, respectively) is $|||e|||_{2p}$ where

$$|||e|||_{2p} = \left[\sum_i \int_{D_i^2} \left[\frac{\partial e}{\partial z}^T \left(A_E\right)_i \frac{\partial e}{\partial Z}\right]^p\right]^{\frac{1}{2p}} \tag{2}$$

FEARS uses C^o first degree elements of bilinear type on curvilinear rectangles (which are mapped into bilinear elements on the master square) and adaptively constructs the meshes by refinement where appropriate.

SAMPLE PROBLEM

Let $\Omega \subset R^2$ be a bounded simply connected region with a boundary $\partial\Omega$. The problem is to find a function U satisfying the biharmonic equation

$$\Delta\Delta U = f \quad \text{on } \Omega \tag{3}$$

and the boundary conditions

$$U = g, \ \frac{\partial U}{\partial n} = h \quad \text{on } \partial\Omega \tag{4}$$

Assume that there is a smooth function Z such that $g=Z$ and $h = \frac{\partial Z}{\partial n}$ on $\partial\Omega$. U is the deflection of the plate. The second derivatives of U are the bending moments which are of special interest. At present only the problem of a built-in plate with boundary conditions of the type given by Equation (4) will be considered.

REDUCTION TO THE LAME' SYSTEM OF ELASTICITY EQUATIONS

Assume that $f=0$ in Equation (3). Then U is a biharmonic function which can be regarded as the Airy function for the plane elasticity problem. Therefore, there exist functions u_1, u_2 defined on Ω such that

$$\begin{aligned} \frac{\partial^2 U}{\partial x_1^2} &= \frac{\partial u_2}{\partial x_2} \\ \frac{\partial^2 U}{\partial x_2^2} &= \frac{\partial u_1}{\partial x_1} \\ -\frac{\partial^2 U}{\partial x_1 \partial x_2} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \end{aligned} \tag{5}$$

and the functions u_1, u_2 also satisfy the well-known Lame' equations (for $\nu = 0$ where ν is Poisson's ratio).

The functions g and h must be transformed into the boundary conditions for u_1, u_2. It is not hard to show that u_1, u_2 satisfy the traction condition on the boundary. However, it must be emphasized that Ω is assumed to be simply connected. (If the plate were free, then the transformed problem would provide the prescribed displacements for u_1, u_2 on $\partial\Omega$.)

The relationship between the traction and the functions g and h is not complicated, but it involves differentiation along the boundary. The assumption that the functions g and h are defined in terms of the function Z simplifies the transformation. The framework of the FEARS program is especially well suited to take advantage of this assumption.

The weak formulation of the problem for the functions u_1, u_2 is a standard one. The functions u_1, u_2 satisfy the variational form of the problem (virtual work condition) and are not subject to any essential boundary conditions.

More precisely, $u_1, u_2 \varepsilon H^1$ (where H^1 is the usual Sobolev space) are functions such that, for $v_1, v_2 \varepsilon H^1$,

$$B_1(u_1,u_2;\ v_1,v_2) = F(v_1,v_2) \tag{6}$$

Here the bilinear form B is the usual form of virtual work obtained by taking $E=1$, $\nu=0$. (E is Young's modulus of elasticity and ν is Poisson's ratio.) Then

$$B_1(u_1,u_2;\ v_1,v_2) = \iint_\Omega \left[\frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\right] dx_1 dx_2 \tag{7}$$

The functional $F(v_1,v_2)$ expresses the virtual work of the traction forces on the boundary. Integration by parts provides $F(v_1,v_2)$ in terms of the function Z. Thus

$$F(v_1,v_2) = \iint_\Omega \left[\frac{\partial v_1}{\partial x_1}\frac{\partial^2 Z}{\partial x_2^2} - \frac{\partial v_2}{\partial x_1}\frac{\partial^2 Z}{\partial x_1 x_2} - \frac{\partial v_1}{\partial x_2}\frac{\partial^2 Z}{\partial x_1 \partial x_2} + \frac{\partial v_2}{\partial x_2}\frac{\partial^2 Z}{\partial x_2^2}\right] dx_1 dx_2 \tag{8}$$

This functional is of a form which the FEARS program can handle if

$$S_1 = \frac{\partial^2 Z}{\partial x_2^2}$$

$$S_2 = S_3 = -\frac{\partial^2 Z}{\partial x_1 \partial x_2}$$

$$S_4 = \frac{\partial^2 Z}{\partial x_1^2}$$

The conditions given by Equations (6), (7), and (8) determine u_1,u_2 uniquely only up to rigid body motion. Accordingly, a displacement is specified at one point and a rotation through another point.

When the functions u_1,u_2 have been obtained, the second derivatives of U (the bending moments) are provided by Equation (5). An additional integration must be performed to obtain the values of U. If however, the bending moments are the main concern, then the functions u_1,u_2 and Equation (5) provide them directly.

So far it has been assumed f=0. Now assume that $f \neq 0$ and g=h=0. In this case a particular solution is needed, which often (e.g., for f=1) can be easily found. If this particular solution is denoted by Z, then

$$U = Z - W$$

where

$$\Delta\Delta W = 0 \text{ on } \Omega$$

and

$$Z = W, \; \frac{\partial Z}{\partial n} = \frac{\partial W}{\partial n} \text{ on } \partial\Omega.$$

This is exactly the case discussed earlier. Now

$$\frac{\partial^2 U}{\partial x_1^i \partial^j x_2} = \frac{\partial^2 Z}{\partial x_1^i \partial x_2^j} - \frac{\partial^2 W}{\partial x_1^i \partial x_2^j}, \; i,j = 0,1,2 \text{ such that } i+j = 2 \tag{9}$$

This approach has the following advantages and disadvantages:

1) Advantages. The finite element method for solving elasticity problems is well-founded theoretically. The error with respect to the energy

norm (which is essentially the L_2 norm for the bending moments) can be reliably estimated. The postprocessing approach allows the computation of the moments at selected points with generally better accuracy along with an error estimate. Likewise the stress intensity factors can also be computed.

2) Disadvantages. If $f \neq 0$, then a particular solution Z must be known. Obtaining the particular solution Z for a general f could be very laborious. However obtaining Z for a particular f in most applied problems is not that difficult. Often Z can be found by inspection or some artifice. Similarly the computation of the displacements (the values of U) could also be quite laborious. In addition, the accuracy of the numerical solution depends on the selection of the particular solution. Since values of the moments are obtained from Equation (9) by subtracting numbers of approximately the same magnitude, the relative error in the computation of W can result in a much larger relative error in $\frac{\partial^2 U}{\partial x_1^i \partial x_2^j}$. Accordingly, it is necessary to compute W with greater relative accuracy. If the domain Ω is not simply connected, then additional difficulties arise.

SPLITTING METHOD

Although the theory of this method has been studied by various authors,[8,9,13,14,15,16] the method is still not very well understood. Today's theoretical results presuppose assumptions which are rather restrictive and it is not clear whether these assumptions are really necessary. These basic assumptions are

(1) The elements used are at least of degree 2.

(2) The mesh is quasi-uniform.

(3) The domain Ω is convex.

However, these assumptions do not seem necessary. Computational results obtained in the course of this investigation strongly suggest that the assumptions are sufficient and not necessary conditions.

For simplicity, consider the problem given by Equations (3), (4) with $g=h=0$. The method consists in writing Equation (3) in the form

$$\begin{aligned} \Delta u_1 &= u_2 \\ -\Delta u_2 &= -f \end{aligned} \tag{10}$$

where the boundary conditions are

$$u_1 = 0 \tag{11}$$

$$\frac{\partial u_1}{\partial n} = 0 \tag{12}$$

The bilinear form is

$$\begin{aligned} &B_2(u_1,u_2;\ v_1,v_2) = \\ &\iint_\Omega \left(\frac{\partial u_1}{\partial x_1}\frac{\partial v_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial v_2}{\partial x_2} + u_2 v_2 \right. \\ &\left. + \frac{\partial u_2}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_1}{\partial x_2} \right) dx_1 dx_2 \end{aligned} \tag{13}$$

and the functional is

$$F_2(v_1,v_2) = -\iint_\Omega f v_1 \ dx_1 dx_2 \tag{14}$$

Then the problem can be written in the standard weak form

$$B_2(u_1,u_2;\ v_1,v_2) = F_2(v_1,v_2) \tag{15}$$

with

$$u_1, v_1 \in \overset{\circ}{H}{}^1,\ u_2, v_2 \in H^1 \text{ where } \overset{\circ}{H}{}^1 = \left\{ u \in H^1 \mid u = 0 \text{ on } \partial\Omega \right\}$$

The boundary condition given by Equation (11) is essential and the boundary condition given by Equation (12) is natural. The bilinear form B_2 is not positive definite and Equation (15) is a "saddle point" problem.

The bilinear form B_2 and the functional F_2 can be used by the FEARS program. However, since FEARS employs elements of degree 1, the first condition is violated. The numerical examples of this investigation also violate the second and third conditions.

There are various options for error norms in the FEARS program. The norms

$$||u||_1^2 = \iint_\Omega \left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right] dx_1 dx_2 \tag{16}$$

or

$$||u||_2^2 = \iint_\Omega \left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right] dx_1 dx_2 \tag{17}$$

can be used. For a nonconvex domain the norm $||u||_1$ of the exact solution u is infinite. If the solution is smooth then either of the norms $||.||_1$ or $||.||_2$ may be used. If the solution has singular behavior, then only the norm $||.||_2$ can be used.

This approach has the following several advantages and disadvantages:

1) Advantages. The problem can be solved by FEARS without obtaining a particular solution. This approach avoids the difficulty stemming from the arbitrary selection of a particular solution and it also avoids the problem of

significance due to subtraction encountered with the previous method. The method directly computes the displacement and its Laplacian. The assumption that the domain is simply connected is not required for this method.

2) Disadvantages. The method is not completely understood theoretically. The saddle point approach does not guarantee monotonic convergence (which occurs when the bilinear form is positive definite). At present the convergence of the method has not been established under the conditions of the FEARS computation. The second derivatives (which are the moments) must be computed by postprocessing. The error norms used by FEARS do not monitor the accuracy of the moments.

NUMERICAL EXPERIMENTATION

Consider the three examples given by Equations (3) and (4) where f=64 on domains Ω_1, Ω_2, Ω_3 and g=h=0 on their boundaries $\partial\Omega_1$, $\partial\Omega_2$, $\partial\Omega_3$.

Example 1 - Uniformly Loaded Circular Plate

$$\Omega_1 = \left\{(x_1,x_2) \mid x_1 + x_2 < 1\right\}$$

Ω_1 is partitioned into the five 2-D domains shown in Figure 1.

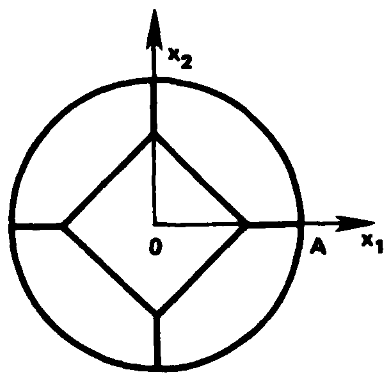

Figure 1 - Partition of Domain Ω_1

The exact solution is known to be

$$U = (1-(x_1 + x_2))^2$$

Example 2 - Uniformly Loaded Square Plate

$$\Omega_2 = \{(x_1,x_2) \mid |x_i| < 1, i = 1,2\}$$

Figure 2 shows the unpartitioned domain Ω_2.

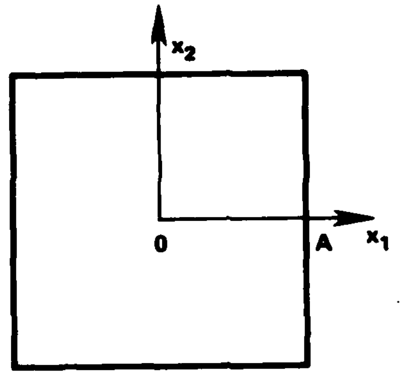

Figure 2 - Domain Ω_2

The solution is not available in closed form though there are analytical expressions for U in series form.

Example 3 - Uniformly Loaded Square Plate With a Slit

$$\Omega_3 = \Omega_2 - \{(x_1,x_2) \mid -1 < x_1 < 0, x_2 = 0\}$$

3 is partitioned into four equal squares as shown in Figure 3.

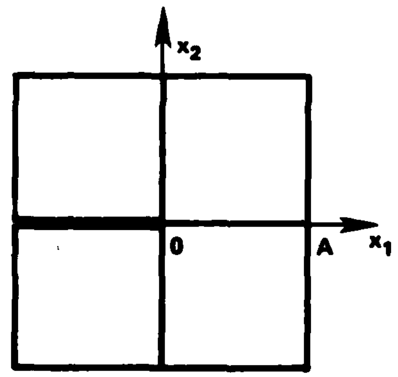

Figure 3 - Partition of Domain Ω_3

The exact solution is not available for Example 3.

The exact solution is very smooth for Example 1, fairly smooth at the corners for Example 2, and has a strong singularity at the origin for Example 3. The trial and test functions are bilinear for Examples 2 and 3, and for Example 1 they are mapped onto bilinear functions. These examples have different axes of symmetry which were utilized to a limited extent, but the material in the tables is organized as though no symmetries were present. The two approaches of the third section will now be discussed.

REDUCTION TO THE LAME' EQUATIONS OF ELASTICITY

We use the following functions Z and S_i.

$$Z^{(1)} = \left(x_1^2 + x_2^2\right)^2$$

$$S_1^{(1)} = 4x_1^2 + 12x_2^2$$

$$S_2^{(1)} = S_3^{(1)} = -8x_1x_2$$

$$S_4^{(1)} = 12x_1^2 + 4x_2^2$$

$$Z^{(2)} = 8/3x_1^4$$

$$S_1^{(2)} = S_2^{(2)} = S_3^{(2)} = 0$$

$$S_4^{(2)} = 32x_1^2$$

$$Z^{(3)} = 8x^2x^2$$

$$S_1^{(3)} = 16x_1^2$$

$$S_2^{(3)} = S_3^{(3)} = -32x_1x_2$$

$$S_4^{(3)} = 16x_2^2$$

The method provides the expected results. The selection of Z barely influenced the relative accuracy of the computed function W with respect to the energy norm when the same number of elements was used. The error estimation was very effective. The main problem of the method is related to the subtraction of the values of Z and W. Table 1 shows typical results for Example 3 ($Z = Z^{(1)}$) with a mesh of 1024 elements. This table displays the values of the second derivatives of the functions Z and W at the point (.53125, .53125).

TABLE 1 - THE VALUES OF THE SECOND PARTIAL DERIVATIVES OF Z AND W AT THE POINT (.53125, .53125) (EXAMPLE 3, $Z = Z^{(1)}$)

	$\frac{\partial^2}{\partial x_1^2}$	$\frac{\partial^2}{\partial x_2^2}$	$\frac{\partial^2}{\partial x_1 \partial x_2}$
W	5.89848	5.72560	1.81340
Z	4.51562	4.51562	2.25781

Now an accuracy of the order of 5% in $\frac{\partial^2 U}{\partial x_i x_j}$, where $U=Z-W$, means that W must have an accuracy in the neighborhood of 1-2%. This restriction on W could be prohibitive. This problem arises in the most important case when $f \neq 0$. If $f=0$, then this problem is usually not so severe.

SPLITTING METHOD

As noted in the previous section, convergence is not theoretically guaranteed for this method. The solution was computed using both a uniform mesh and an adaptive mesh with respect to the error estimator of the $||\cdot||_2$ norm given by Equation (17). The results are displayed in Tables 2-5. In these tables the computed values of u_1 and u_2 on the line OA are given. (As the meshes are refined, more values on OA are computed.) The headings of these tables are as follows:

(1) The x_1- coordinate of the point $(x_1,0)$
(2) The computed value of $u_1(x_1,0)$
(3) The computed value of $u_2(x_1,0)$

TABLE 2 - EXAMPLE 1: UNIFORM MESH

N	80		320		1280		Exact	
x_1	$u_1(x_1,0)$	$u_2(x_1,0)$	$u_1(x_1,0)$	$u_2(x_1,0)$	$u_1(x_1,0)$	$u_2(x_1 0)$	$u_1(x_1,0)$	$u_2(x_1;0)$
0	1.00497	-8.15913	1.00127	-8.04037	1.00032	-8.01013	1.000000	-8.000000
.06250					.992521	-7.94763	.992203	-7.93750
.12500			.970230	-7.79035	.969305	-7.76012	.968994	-7.75000
.18750					.931220	-7.44757	.930923	-7.43750
.25000	.883213	-7.15635	.879993	-7.03965	.879173	-7.00995	.878906	-7.00000
.31250					.814460	-6.44717	.814224	-6.43750
.37500			.739237	-5.78591	.738704	-5.75909	.738525	-5.75000
.43750					.653912	-4.94525	.653824	-4.93750
.50000	.562680	-4.09640	.562265	-4.01785	.562365	-4.00290	.562500	-4.00000
.53125					.515320	-3.49129	.51599	-3.48438
.56250			.467998	-2.96921	.467483	-2.94548	.467300	-2.93750
.59375					.419387	-2.36741	.419206	-2.35938
.62500	.374055	-1.87763	.371971	-1.78068	.371494	-1.75762	.371338	-1.75000
.65625					.324264	-1.11629	.324143	-1.10938
.68750			.278453	-.461889	.278175	-.443470	.278091	-.437500
.71875					.233721	.260799	.233674	.265625
.75000	.192019	.931318	.191492	.985424	.191421	.996522	.191406	1.00000
.78125					.151813	1.76373	.151826	1.76563
.81250			.115372	2.56157	.115460	2.56247	.115494	2.56250
.84375					.0829456	3.39288	.0829935	3.39063
.87500	.0540231	4.29448	.0547048	4.26842	.0548768	4.25515	.0549316	4.25000
.90625					.0318831	5.14975	.031975	5.14063
.93750			.0144603	6.11049	.0146170	6.07757	.0146637	6.06250
.96875					.00375493	7.04051	.00378513	7.01563
1.0000	0	8.23284	0	8.09769	0	8.04265	0	8.00000

TABLE 3 - EXAMPLE 2: UNIFORM MESH

N	64		256		1024	
x_1	$u_1(x_1,0)$	$u_2(x_1,0)$	$u_1(x_1,0)$	$u_2(x_1 0)$	$u_1(x_1 0)$	$u_1(x_1,0)$
0	1.35644	-9.61960	1.31121	-9.17025	1.29959	-9.05834
.0625					1.29076	-8.99565
.1250			1.27571	-8.91722	1.26440	-8.80644
.1875					1.22086	-8.48732
.2500	1.21137	-8.57152	1.17114	-8.13994	1.16078	-8.03261
.3125					1.08506	-7.43436
.3750			1.00384	-6.78387	.994969	-6.68236
.4375					.892184	-5.76413
.5000	.812358	-5.13491	.785777	-4.75807	.778854	-4.66495
.5625					.657709	-3.36783
.6250			.536873	-1.93536	.532167	-1.85355
.6875					.406453	-1.00715
.7500	.297489	1.57309	.288256	1.84719	.285755	1.91421
.8125					.176378	4.21680
.8750			.0866609	6.78649	.0859307	6.83454
.9375					.0235262	9.79657
1,0000	0	13.0037	0	13.1088	0	13.1334

TABLE 4 - EXAMPLE 3: UNIFORM MESH

N	64		256		1024	
x_1	$u_1(x_1 0)$	$u_2(x_1,0)$	$u_1(x_1 0)$	$u_2(x_1,0)$	$u_1(x_1,0)$	$u_2(x_1,0)$
0	0	14.0294	0	20.5651	0	29.0652
.0625					.0366752	9.02679
.1250			.097985	3.21982	.0856739	4.39889
.1875					.133943	1.45009
.2500	.199010	-1.94006	.185872	-.756482	.176257	-.427232
.3125					.209127	-1.66725
.3750			.239824	-2.66616	.230487	-2.44152
.4375					.239298	-2.84153
.5000	.259715	-3.50934	.243199	-3.06997	.235401	-2.91838
.5625					.219446	-2.70160
.6250			.198364	-2.30707	.192882	-2.20784
.6875					.157971	-1.44573
.7500	.127328	-.718524	.120795	-.468820	.117826	-.418792
.8125					.0764653	.872537
.8750			.0397549	2.43183	.0388672	2.42922
.9375					.0110272	4.25203
1	0	6.41809	0	6.40323	0	6.33992

TABLE 5 - EXAMPLE 3: THE ADAPTIVE MESH

N	64		208		634	
x_1	$u(x_1,0)$	$u_2(x_1,0)$	$u_1(x_1,0)$	$u_2(x_1,0)$	$u_1(x_1,0)$	$u_2(x_1,0)$
0	0	14.0294	0	20.5741	0	29.0727
.0625					.0366943	9.02608
.1250			.0918232	3.21720	.0857160	4.39572
.1875					.134002	1.44502
.2500	.199010	-1.94006	.185894	-.760749	.176321	-.433832
.3125					.209179	-1.67517
.3750			.239801	-2.67162	.230509	-2.45064
.4375					.239270	-2.85178
.5000	.259715	-3.50934	.243121	-3.07583	.235305	-2.92978
.5625					.219247	-2.71465
.6250			.198263	-2.31303	.192620	-2.21778
.7500	.127328	-.718524	.120724	-.475214	.117593	-.429559
.8125					.0762835	.858996
.8750			.039362	2.42498	.0387926	2.41697
.9375					.0110087	4.24102
1	0	6.41809	0	6.40290	0	6.33114

(4) The exact value of $u_1(x_1,0)$ (when available)

(5) The exact value of $u_2(x_1,0)$ (when available)

(6) Number of elements N on the full domain (without using symmetry)

Table 2 shows the results for Example 1. In this case the exact solution is known. Good results have been obtained especially with respect to the value of u_1 (the displacement). The accuracy of the function u_2 is also good, but not quite as good as that of u_1. Because the solution is smooth, the adaptive mesh is uniform.

Table 3 shows the results for Example 2. The solution is smooth so the adaptively constructed mesh is uniform. The exact solution is not available, but the results improve as the number of elements increases. As before, the function u_1 is more accurate than the function u_2. This is to be expected since $u_2 = \Delta u_1$.

Table 4 shows the results for Example 3 when a uniform non-adaptive mesh is used. The exact solution has a strong singularity at the origin and $u_2(x_1,0) \to \infty$ as $x_1 \to 0$. The exact solution is not available. Once more the results are good especially with respect to u_1.

Table 5 gives the results for Example 3 when an adaptive mesh is used. Again the method provides good results. The accuracy for an adaptive mesh of 634 elements is comparable to the accuracy provided by a uniform mesh of 1024 elements.

The results displayed in these tables are typical. Other test computations were performed which tend to confirm the general behavior shown in these tables.

CONCLUSIONS

Although convergence of the splitting method is theoretically guaranteed at present only under fairly stringent assumptions, it seems to work very well also when these assumptions are not satisfied. The method is a mixed one of the "saddle point" type. Its performance is very good with respect to the displacement u_1 and slightly worse with respect to u_2 or u_1. It is apparent that the method is reliable, particularly if it is controlled by an error estimator. Accordingly it is recommended for use.

A reliable extraction of the moments and shear forces from the computed results would appear to require postprocessing. This problem should be addressed. It is hoped that the convergence of the method will be theoretically established under less stringent assumptions that those imposed at present and will apply to the examples tested in this report. The splitting method is preferable to the reduction to the Lame' elasticity equations.

The method based on the reduction to the Lame' elasticity equations is very well founded theoretically, but practically has some essential disadvantages which makes it less attractive. The main disadvantages of this method are:

- The need for a particular solution.
- The loss of accuracy and increased error due to the subtraction of the particular solution.
- The necessity of assuming that the domain is simply connected. (This restriction simplifies the computational effort considerably.)
- The method does not directly provide the displacements.

It would appear that the splitting method is the preferred method for the FEARS user.

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